

## ON SOME PARTITIONS OF A FLAG MANIFOLD

G. LUSZTIG

**1.** Let  $G$  be a connected reductive group over an algebraically closed field  $\mathbf{k}$  of characteristic exponent  $p \geq 1$ . We shall assume that  $p$  is 1 or a good prime for  $G$ . Let  $\mathcal{B}$  be the variety of Borel subgroups of  $G$  and let  $\mathbf{W}$  be the Weyl group of  $G$ . Note that  $\mathbf{W}$  naturally indexes ( $w \mapsto \mathcal{O}_w$ ) the orbits of  $G$  acting on  $\mathcal{B} \times \mathcal{B}$  by simultaneous conjugation on the two factors. For  $g \in G$  we set  $\mathcal{B}_g = \{B \in \mathcal{B}; g \in B\}$ . The varieties  $\mathcal{B}_g$  play an important role in representation theory and their geometry has been studied extensively. More generally for  $g \in G$  and  $w \in \mathbf{W}$  we set

$$\mathcal{B}_g^w = \{B \in \mathcal{B}; (B, gBg^{-1}) \in \mathcal{O}_w\}.$$

Note that  $\mathcal{B}_g^1 = \mathcal{B}_g$  and that for fixed  $g$ ,  $(\mathcal{B}_g^w)_{w \in \mathbf{W}}$  form a partition of the flag manifold  $\mathcal{B}$ .

For fixed  $w$ , the varieties  $\mathcal{B}_g^w$  ( $g \in G$ ) appear as fibres of a map to  $G$  which was introduced in [L3] as part of the definition of character sheaves. Earlier, the varieties  $\mathcal{B}_g^w$  for  $g$  regular semisimple appeared in [L1] (a precursor of [L3]) where it was shown that from their topology (for  $\mathbf{k} = \mathbf{C}$ ) one can extract nontrivial information about the character table of the corresponding group over a finite field.

In this paper we present some experimental evidence for our belief that from the topology of  $\mathcal{B}_g^w$  for  $g$  unipotent (with  $\mathbf{k} = \mathbf{C}$ ) one can extract information closely connected with the map [KL, 9.1] from unipotent classes in  $G$  to conjugacy classes in  $\mathbf{W}$ .

I thank David Vogan for some useful discussions.

**2.** We fix a prime number  $l$  invertible in  $\mathbf{k}$ . Let  $g \in G$  and  $w \in \mathbf{W}$ . For  $i, j \in \mathbf{Z}$  let  $H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)_j$  be the subquotient of pure weight  $j$  of the  $l$ -adic cohomology space  $H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)$ . The centralizer  $Z(g)$  of  $g$  in  $G$  acts on  $\mathcal{B}_g^w$  by conjugation and this induces an action of the group of components  $\bar{Z}(g)$  on  $H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)$  and on each  $H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)_j$ . For  $z \in \bar{Z}(g)$  we set

$$\Xi_{g,z}^w = \sum_{i,j \in \mathbf{Z}} (-1)^i \text{tr}(z, H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)_j) v^j \in \mathbf{Z}[v]$$

---

Supported in part by the National Science Foundation

where  $v$  is an indeterminate; the fact that this belongs to  $\mathbf{Z}[v]$  and is independent of the choice of  $l$  is proved by an argument similar to that in the proof of [DL, 3.3].

Let  $l : \mathbf{W} \rightarrow \mathbf{N}$  be the standard length function. The simple reflections  $s \in \mathbf{W}$  (that is the elements of length 1 of  $\mathbf{W}$ ) are numbered as  $s_1, s_2, \dots$ . Let  $w_0$  be the element of maximal length in  $\mathbf{W}$ . Let  $\underline{\mathbf{W}}$  be the set of conjugacy classes in  $\mathbf{W}$ .

Let  $\mathcal{H}$  be the Iwahori-Hecke algebra of  $\mathbf{W}$  with parameter  $v^2$  (see [GP, 4.4.1]; in the definition in *loc.cit.* we take  $A = \mathbf{Z}[v, v^{-1}]$ ,  $a_s = b_s = v^2$ ). Let  $(T_w)_{w \in \mathbf{W}}$  be the standard basis of  $\mathcal{H}$  (see [GP, 4.4.3, 4.4.6]). For  $w \in \mathbf{W}$  let  $\hat{T} = v^{-2l(w)}T_w$ . If  $s_{i_1}s_{i_2}\dots s_{i_t}$  is a reduced expression for  $w \in \mathbf{W}$  we write also  $\hat{T}_w = \hat{T}_{i_1i_2\dots i_t}$ .

For any  $g \in G$ ,  $z \in \bar{Z}(g)$  we set

$$\Pi_{g,z} = \sum_{w \in \mathbf{W}} \Xi_{g,z}^w \hat{T}_w \in \mathcal{H}.$$

The following result can be proved along the lines of the proof of [DL, Theorem 1.6] (we replace the Frobenius map in that proof by conjugation by  $g$ ); alternatively, for  $g$  unipotent, we may use 6(a).

(a)  $\Pi_{g,z}$  belongs to the centre of the algebra  $\mathcal{H}$ .

According to [GP, 8.2.6, 7.1.7], an element  $c = \sum_{w \in \mathbf{W}} c_w \hat{T}_w$  ( $c_w \in \mathbf{Z}[v, v^{-1}]$ ) in the centre of  $\mathcal{H}$  is uniquely determined by the coefficients  $c_w$  ( $w \in \mathbf{W}_{\min}$ ) and we have  $c_w = c_{w'}$  if  $w, w' \in \mathbf{W}_{\min}$  are conjugate in  $\mathbf{W}$ ; here  $\mathbf{W}_{\min}$  is the set of elements of  $\mathbf{W}$  which have minimal length in their conjugacy class. This applies in particular to  $c = \Pi_{g,z}$ , see (a). For any  $C \in \underline{\mathbf{W}}$  we set  $\Xi_{g,z}^C = \Xi_{g,z}^w$  where  $w$  is any element of  $C \cap \mathbf{W}_{\min}$ .

Note that if  $g = 1$  then  $\Pi_{g,1} = (\sum_w v^{2l(w)}1)$ . If  $g$  is regular unipotent then  $\Pi_{g,1} = \sum_{w \in \mathbf{W}} v^{2l(w)}\hat{T}_w$ . If  $G = PGL_3(\mathbf{k})$  and  $g \in G$  is regular semisimple then  $\Pi_{g,1} = 6 + 3(v^2 - 1)(\hat{T}_1 + \hat{T}_2) + (v^2 - 1)^2(\hat{T}_{12} + \hat{T}_{21}) + (v^6 - 1)\hat{T}_{121}$ ; if  $g \in G$  is a transvection then  $\Pi_{g,1} = (2v^2 + 1) + v^4(\hat{T}_1 + \hat{T}_2) + v^6\hat{T}_{121}$ .

For  $g \in G$  let  $cl(g)$  be the  $G$ -conjugacy class of  $g$ ; let  $\overline{cl(g)}$  be the closure of  $cl(g)$ . Let  $\mathbf{S}_g$  be the set of all  $C \in \underline{\mathbf{W}}$  such that  $\Xi_{g,1}^C \neq 0$  and  $\Xi_{g',1}^C = 0$  for any  $g' \in \overline{cl(g)} - cl(g)$ . If  $\mathcal{C}$  is a conjugacy class in  $G$  we shall also write  $\mathbf{S}_{\mathcal{C}}$  instead of  $\mathbf{S}_g$  where  $g \in \mathcal{C}$ .

We describe the set  $\mathbf{S}_g$  and the values  $\Xi_{g,1}^C$  for  $C \in \mathbf{S}_g$  for various  $G$  of low rank and various unipotent elements  $g$  in  $G$ . We denote by  $u_n$  a unipotent element of  $G$  such that  $\dim \mathcal{B}_{u_n} = n$ . The conjugacy class of  $w \in \mathbf{W}$  is denoted by  $(w)$ .

$G$  of type  $A_1$ .

$$\mathbf{S}_{u_1} = (1), \mathbf{S}_{u_0} = (s_1); \Xi_{u_1,1}^1 = 1 + v^2, \Xi_{u_0,1}^{s_1} = v^2.$$

$G$  of type  $A_2$ .

$$\mathbf{S}_{u_3} = (1), \mathbf{S}_{u_1} = (s_1), \mathbf{S}_{u_0} = (s_1s_2).$$

$$\Xi_{u_3,1}^1 = 1 + 2v^2 + 2v^4 + v^6, \Xi_{u_1,1}^{(s_1)} = v^4, \Xi_{u_0,1}^{(s_1 s_2)} = v^4.$$

$G$  of type  $B_2$ . (The simple reflection corresponding to the long root is denoted by  $s_1$ .)

$$\mathbf{S}_{u_4} = (1), \mathbf{S}_{u_2} = (s_1), \mathbf{S}_{u_1} = \{(s_2), (s_1 s_2 s_1 s_2)\}, \mathbf{S}_{u_0} = (s_1 s_2).$$

$$\begin{aligned} \Xi_{u_4,1}^1 &= (1 + v^2)^2(1 + v^4), \Xi_{u_2,1}^{(s_1)} = v^4(1 + v^2), \Xi_{u_1,1}^{(s_2)} = 2v^4, \\ \Xi_{u_1,1}^{(s_1 s_2 s_1 s_2)} &= v^6(v^2 - 1), \Xi_{u_0,1}^{(s_1 s_2)} = v^4. \end{aligned}$$

$G$  of type  $G_2$ . (The simple reflection corresponding to the long root is denoted by  $s_2$ .)

$$\mathbf{S}_{u_6} = (1), \mathbf{S}_{u_3} = (s_2), \mathbf{S}_{u_2} = \{(s_1), (s_1 s_2 s_1 s_2 s_1 s_2)\}, \mathbf{S}_{u_1} = (s_1 s_2 s_1 s_2), \mathbf{S}_{u_0} = (s_1 s_2). \blacksquare$$

$$\begin{aligned} \Xi_{u_6,1}^1 &= (1 + v^2)^2(1 + v^4 + v^8), \Xi_{u_3,1}^{(s_2)} = v^6(1 + v^2), \Xi_{u_2,1}^{(s_1)} = v^4(1 + v^2), \\ \Xi_{u_2,1}^{(s_1 s_2 s_1 s_2 s_1 s_2)} &= v^8(v^4 - 1), \Xi_{u_1,1}^{(s_1 s_2 s_1 s_2)} = 2v^8, \Xi_{u_0,1}^{(s_1 s_2)} = v^4. \end{aligned}$$

$G$  of type  $B_3$ . (The simple reflection corresponding to the short root is denoted by  $s_3$  and  $(s_1 s_3)^2 = 1$ .)

$$\begin{aligned} \mathbf{S}_{u_9} &= (1), \mathbf{S}_{u_5} = (s_1), \mathbf{S}_{u_4} = \{(s_3), (s_2 s_3 s_2 s_3)\}, \mathbf{S}_{u_3} = \{(s_1 s_3), (w_0)\}, \\ \mathbf{S}_{u_2} &= (s_1 s_2), \mathbf{S}_{u_1} = \{(s_2 s_3), (s_2 s_3 s_1 s_2 s_3)\}, \mathbf{S}_{u_0} = (s_1 s_2 s_3). \end{aligned}$$

$$\begin{aligned} \Xi_{u_9,1}^1 &= (1 + v^2)^3(1 + v^4)(1 + v^4 + v^8), \Xi_{u_5,1}^{(s_1)} = v^8(1 + v^2)^2, \\ \Xi_{u_4,1}^{(s_2 s_3 s_2 s_3)} &= v^8(1 + v^2)(v^4 - 1), \Xi_{u_4,1}^{(s_3)} = 2v^6(1 + v^2)^2, \\ \Xi_{u_3,1}^{(s_1 s_3)} &= v^8(1 + v^2), \Xi_{u_3,1}^{(w_0)} = v^{14}(v^4 - 1), \Xi_{u_2,1}^{(s_1 s_2)} = 2v^8, \\ \Xi_{u_1,1}^{(s_2 s_3)} &= 2v^6, \Xi_{u_1,1}^{(s_2 s_3 s_1 s_2 s_3)} = v^8(v^2 - 1), \Xi_{u_0,1}^{(s_1 s_2 s_3)} = v^6. \end{aligned}$$

$G$  of type  $C_3$ . (The simple reflection corresponding to the long root is denoted by  $s_3$  and  $(s_1 s_3)^2 = 1$ ;  $u_2''$  denotes a unipotent element which is regular inside a Levi subgroup of type  $C_2$ ;  $u_2'$  denotes a unipotent element with  $\dim \mathcal{B}_{u_2''} = 2$  which is not conjugate to  $u_2''$ .)

$$\begin{aligned} \mathbf{S}_{u_9} &= (1), \mathbf{S}_{u_6} = (s_3), \mathbf{S}_{u_4} = \{(s_1), (s_2 s_3 s_2 s_3)\}, \mathbf{S}_{u_3} = \{(s_1 s_3), (w_0)\}, \\ \mathbf{S}_{u_2'} &= (s_1 s_2), \mathbf{S}_{u_2''} = (s_2 s_3), \mathbf{S}_{u_1} = (s_2 s_3 s_1 s_2 s_3), \mathbf{S}_{u_0} = (s_1 s_2 s_3). \end{aligned}$$

$$\begin{aligned} \Xi_{u_9,1}^1 &= (1 + v^2)^3(1 + v^4)(1 + v^4 + v^8), \Xi_{u_6,1}^{(s_3)} = v^6(1 + v^2)^2(1 + v^4), \\ \Xi_{u_4,1}^{(s_2 s_3 s_2 s_3)} &= v^{10}(v^4 - 1), \Xi_{u_4,1}^{(s_1)} = 2v^8(1 + v^2), \\ \Xi_{u_3,1}^{(s_1 s_3)} &= v^8(1 + v^2), \Xi_{u_3,1}^{(w_0)} = v^{14}(v^4 - 1), \Xi_{u_2',1}^{(s_1 s_2)} = v^6(1 + v^2), \\ \Xi_{u_2'',1}^{(s_2 s_3)} &= v^6(1 + v^2), \Xi_{u_1,1}^{(s_2 s_3 s_1 s_2 s_3)} = v^{10}, \Xi_{u_0,1}^{(s_1 s_2 s_3)} = v^6. \end{aligned}$$

**3.** Consider the following properties of  $G$ :

(a)  $\mathbf{W} = \sqcup_u \mathbf{S}_u$

( $u$  runs over a set of representatives for the unipotent classes in  $G$ );

(b)  $c_u \in \mathbf{S}_u$

where for any unipotent element  $u \in G$ ,  $c_u$  denotes the conjugacy class in  $\mathbf{W}$  associated to  $u$  in [KL, 9.1]. Note that (a),(b) hold for  $G$  of rank  $\leq 3$  (see §2). We have used the computations of the map in [KL, 9.1] given in [KL, §9], [S1], [S2]. We will show elsewhere that (a), (b) hold for  $G$  of type  $A_n$  or  $C_n$ . We expect that a proof similar to that in type  $C_n$  would yield (a),(b) in type  $B_n$  and  $D_n$ . We expect that (a),(b) hold also for  $G$  simple of exceptional type; (a) should follow by computing the product of some known (large) matrices using 6(a). The equality  $\mathbf{W} = \cup_u \mathbf{S}_u$  is clear since for a regular unipotent  $u$  and any  $w$  we have  $\Xi_{u,1}^w = v^{2l(w)}$ .

**4.** Assume that  $G = Sp_{2n}(\mathbf{k})$ . The Weyl group  $\mathbf{W}$  can be identified in the standard way with the subgroup of the symmetric group  $S_{2n}$  consisting of all permutations of  $\{1, 2, \dots, n, n', \dots, 2', 1'\}$  which commute with the involution  $1 \leftrightarrow 1', 2 \leftrightarrow 2', \dots, n \leftrightarrow n'$ . We say that two elements of  $\mathbf{W}$  are equivalent if they are contained in the same conjugacy class of  $S_{2n}$ . The set of equivalence classes in  $\mathbf{W}$  is in bijection with the set of partitions of  $2n$  in which every odd part appears an even number of times (to  $C \in \mathbf{W}$  we attach the partition which has a part  $j$  for every  $j$ -cycle of an element of  $C$  viewed as a permutation of  $\{1, 2, \dots, n, n', \dots, 2', 1'\}$ ). The same set of partitions of  $2n$  indexes the set of unipotent classes of  $G$ . Thus we obtain a bijection between the set of equivalence classes in  $\mathbf{W}$  and the set of unipotent classes of  $G$ . In other words we obtain a surjective map  $\phi$  from  $\mathbf{W}$  to the set of unipotent classes of  $G$  whose fibres are the equivalence classes in  $\mathbf{W}$ . One can show that for any unipotent class  $\mathcal{C}$  in  $G$  we have  $\phi^{-1}(\mathcal{C}) = \mathbf{S}_u$  where  $u \in \mathcal{C}$ .

**5.** Recall that the set of unipotent elements in  $G$  can be partitioned into "special pieces" (see [L5]) where each special piece is a union of unipotent classes exactly one of which is "special". Thus the special pieces can be indexed by the set of isomorphism classes of special representations of  $\mathbf{W}$  which depends only on  $\mathbf{W}$  as a Coxeter group (not on the underlying root system). For each special piece  $\sigma$  of  $G$  we consider the subset  $\mathbf{S}_\sigma := \sqcup_{\mathcal{C} \subset \sigma} \mathbf{S}_\mathcal{C}$  of  $\mathbf{W}$  (here  $\mathcal{C}$  runs over the unipotent classes contained in  $\sigma$ ). We expect that each such subset  $\mathbf{S}_\sigma$  depends only on the Coxeter group structure of  $\mathbf{W}$  (not on the underlying root system). As evidence for this we note that the subsets  $\mathbf{S}_\sigma$  for  $G$  of type  $B_3$  are the same as the subsets  $\mathbf{S}_\sigma$  for  $G$  of type  $C_3$ . These subsets are as follows:

$$\begin{aligned} &\{1\}, \{(s_1), (s_3), (s_2s_3s_2s_3)\}, \{(s_1s_3), (w_0)\}, \{(s_1s_2)\}, \\ &\{(s_2s_3), (s_2s_3s_1s_2s_3)\}, \{(s_1s_2s_3)\}. \end{aligned}$$

**6.** Let  $g \in G$  be a unipotent element and let  $z \in \bar{Z}(g)$ ,  $w \in W$ . We show how the polynomial  $\Xi_{g,z}^w$  can be computed using information from representation theory. We may assume that  $p > 1$  and that  $\mathbf{k}$  is the algebraic closure of the finite field  $\mathbf{F}_p$ . We choose an  $\mathbf{F}_p$  split rational structure on  $G$  with Frobenius map  $F_0 : G \rightarrow G$ . We may assume that  $g \in G^{F_0}$ . Let  $q = p^m$  where  $m \geq 1$  is sufficiently divisible. In particular  $F := F_0^m$  acts trivially on  $\bar{Z}(g)$  hence  $cl(g)^F$  is a union of  $G^F$ -conjugacy classes naturally indexed by the conjugacy classes in  $\bar{Z}(g)$ ; in particular the  $G^F$ -conjugacy class of  $g$  corresponds to  $1 \in \bar{Z}(g)$ . Let  $g_z$  be an element of the  $G^F$ -conjugacy class in  $cl(g)^F$  corresponding to the  $\bar{Z}(g)$ -conjugacy class of  $z \in \bar{Z}(g)$ . The set  $\mathcal{B}_{g_z}^w$  is  $F$ -stable. We first compute the number of fixed points  $|(\mathcal{B}_{g_z}^w)^F|$ .

Let  $\mathcal{H}_q = \bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}[v, v^{-1}]} \mathcal{H}$  where  $\bar{\mathbf{Q}}_l$  is regarded as a  $\mathbf{Z}[v, v^{-1}]$ -algebra with  $v$  acting as multiplication by  $\sqrt{q}$ . We write  $T_w$  instead of  $1 \otimes T_w$ . Let  $\text{Irr} \mathbf{W}$  be a set of representatives for the isomorphism classes of irreducible  $\mathbf{W}$ -modules over  $\bar{\mathbf{Q}}_l$ . For any  $E \in \text{Irr} \mathbf{W}$  let  $E_q$  be the irreducible  $\mathcal{H}_q$ -module corresponding naturally to  $E$ . Let  $\mathcal{F}$  be the vector space of functions  $\mathcal{B}^F \rightarrow \bar{\mathbf{Q}}_l$ . We regard  $\mathcal{F}$  as a  $G^F$ -module by  $\gamma : f \mapsto f'$ ,  $f'(B) = f(\gamma^{-1}B\gamma)$  for all  $B \in \mathcal{B}^F$ . We identify  $\mathcal{H}_q$  with the algebra of all endomorphisms of  $\mathcal{F}$  which commute with the  $G^F$ -action, by identifying  $T_w$  with the endomorphism  $f \mapsto f'$  where  $f'(B) = \sum_{B' \in \mathcal{B}^F; (B, B') \in \mathcal{O}_w} f(B)$  for all  $B \in \mathcal{B}^F$ . As a module over  $\bar{\mathbf{Q}}_l[G^F] \otimes \mathcal{H}_q$  we have canonically  $\mathcal{F} = \bigoplus_{E \in \text{Irr} \mathbf{W}} \rho_E \otimes E_q$  where  $\rho_E$  is an irreducible  $G^F$ -module. Hence if  $\gamma \in G^F$  and  $w \in \mathbf{W}$  we have  $\text{tr}(\gamma T_w, \mathcal{F}) = \sum_{E \in \text{Irr} \mathbf{W}} \text{tr}(\gamma, \rho_E) \text{tr}(T_w, E_q)$ . From the definition we have  $\text{tr}(\gamma T_w, \mathcal{F}) = |\{B \in \mathcal{B}^F; (B, \gamma B \gamma^{-1}) \in \mathcal{O}_w\}| = |(\mathcal{B}_{\gamma}^w)^F|$ . Taking  $\gamma = g_z$  we obtain

$$(a) \quad |(\mathcal{B}_{g_z}^w)^F| = \sum_{E \in \text{Irr} \mathbf{W}} \text{tr}(g_z, \rho_E) \text{tr}(T_w, E_q).$$

The quantity  $\text{tr}(g_z, \rho_E)$  can be computed explicitly, by the method of [L4], in terms of generalized Green functions and of the entries of the non-abelian Fourier transform matrices [L2]; in particular it is a polynomial with rational coefficients in  $\sqrt{q}$ . The quantity  $\text{tr}(T_w, E_q)$  can be also computed explicitly (see [GP], Ch.10,11); it is a polynomial with integer coefficients in  $\sqrt{q}$ . Thus  $|(\mathcal{B}_{g_z}^w)^F|$  is an explicitly computable polynomial with rational coefficients in  $\sqrt{q}$ . Substituting here  $\sqrt{q}$  by  $v$  we obtain the polynomial  $\Xi_{g,z}^w$ . This argument shows also that  $\Xi_{g,z}^w$  is independent of  $p$  (note that the pairs  $(g, z)$  up to conjugacy may be parametrized by a set independent of  $p$ ).

This is how the various  $\Xi_{g,z}^w$  in §2 were computed, except in type  $A_1, A_2, B_2$  where they were computed directly from the definitions. (For type  $B_3, C_3$  we have used the computation of Green functions in [Sh]; for type  $G_2$  we have used directly [CR] for the character of  $\rho_E$  at unipotent elements.)

**7.** In this section we assume that  $G$  is simply connected. Let  $\tilde{G} = G(\mathbf{k}((\epsilon)))$  where  $\epsilon$  is an indeterminate. Let  $\tilde{\mathcal{B}}$  be the set of Iwahori subgroups of  $\tilde{G}$ . Let  $\tilde{\mathbf{W}}$  the

affine Weyl group attached to  $\tilde{G}$ . Note that  $\tilde{\mathbf{W}}$  naturally indexes ( $w \mapsto \mathcal{O}_w$ ) the orbits of  $\tilde{G}$  acting on  $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$  by simultaneous conjugation on the two factors. For  $g \in \tilde{G}$  and  $w \in \tilde{\mathbf{W}}$  we set

$$\tilde{\mathcal{B}}_g^w = \{B \in \tilde{\mathcal{B}}; (B, gBg^{-1}) \in \mathcal{O}_w\}.$$

By analogy with [KL, §3] we expect that when  $g$  is regular semisimple,  $\tilde{\mathcal{B}}_g^w$  has a natural structure of a locally finite union of algebraic varieties over  $\mathbf{k}$  of bounded dimension and that, moreover, if  $g$  is also elliptic, then  $\tilde{\mathcal{B}}_g^w$  has a natural structure of algebraic variety over  $\mathbf{k}$ . It would follow that for  $g$  elliptic and  $w \in \tilde{\mathbf{W}}$ ,

$$\Xi_g^w = \sum_{i,j \in \mathbf{Z}} (-1)^i \dim H_c^i(\tilde{\mathcal{B}}_g^w, \bar{\mathbf{Q}}_l)_j v^j \in \mathbf{Z}[v]$$

is well defined; one can then show that the formal sum  $\sum_{w \in \tilde{\mathbf{W}}} \Xi_g^w \hat{T}_w$  is central in the completion of the affine Hecke algebra consisting of all formal sums  $\sum_{w \in \tilde{\mathbf{W}}} a_w \hat{T}_w$  ( $a_w \in \mathbf{Q}(v)$ ) that is, it commutes with any  $\hat{T}_w$ . (Here  $\hat{T}_w$  is defined as in §2 and the completion of the affine Hecke algebra is regarded as a bimodule over the actual affine Hecke algebra in the natural way.)

**8.** In this and the next subsection we assume that  $G$  is adjoint. For  $g \in G, z \in \bar{Z}(g), w \in \mathbf{W}$  we set

$$\xi_{g,z}^w = \Xi_{g,z}^w|_{v=1} = \sum_{i \in \mathbf{Z}} (-1)^i \text{tr}(z, H_c^i(\mathcal{B}_g^w, \bar{\mathbf{Q}}_l)) \in \mathbf{Z}.$$

This integer is independent of  $l$ . For any  $g \in G, z \in \bar{Z}(g)$  we set

$$\pi_{g,z} = \sum_{w \in \mathbf{W}} \xi_{g,z}^w w \in \mathbf{Z}[W].$$

This is the specialization of  $\Pi_{g,z}$  for  $v = 1$ . Hence from 2(a) we see that  $\pi_{g,z}$  is in the centre of the ring  $\mathbf{Z}[\mathbf{W}]$ . Thus for any  $C \in \underline{\mathbf{W}}$  we can set  $\xi_{g,z}^C = \xi_{g,z}^w$  where  $w$  is any element of  $C$ . For  $g \in G$  let  $\mathbf{s}_g$  be the set of all  $C \in \underline{\mathbf{W}}$  such that  $\xi_{g,z}^C \neq 0$  for some  $z \in \bar{Z}(g)$  and  $\xi_{g',z'}^C = 0$  for any  $g' \in \overline{cl(g)} - cl(g)$  and any  $z' \in \bar{Z}(g')$ . We describe the set  $\mathbf{s}_g$  and the values  $\xi_{g,z}^C = 0$  for  $C \in \mathbf{s}_g, z \in \bar{Z}(g)$ , for various  $G$  of low rank and various unipotent elements  $g$  in  $G$ . We use the notation in §2. Moreover in the case where  $\bar{Z}(g) \neq \{1\}$  we denote by  $z_n$  an element of order  $n$  in  $\bar{Z}(g)$ .

$G$  of type  $A_1$ .

$$\mathbf{s}_{u_1} = (1), \mathbf{s}_{u_0} = (s_1); \xi_{u_1,1}^1 = 2, \xi_{u_0,1}^{s_1} = 1.$$

$G$  of type  $A_2$ .

$$\mathbf{s}_{u_3} = (1), \mathbf{s}_{u_1} = (s_1), \mathbf{s}_{u_0} = (s_1 s_2).$$

$$\xi_{u_3,1}^1 = 6, \xi_{u_1,1}^{(s_1)} = 1, \xi_{u_0,1}^{(s_1 s_2)} = 1.$$

$G$  of type  $B_2$ .

$$\mathbf{s}_{u_4} = (1), \mathbf{s}_{u_2} = (s_1), \mathbf{s}_{u_1} = \{(s_2), (s_1 s_2 s_1 s_2)\}, \mathbf{s}_{u_0} = (s_1 s_2).$$

$$\begin{aligned} \xi_{u_4,1}^1 &= 8, \xi_{u_2,1}^{(s_1)} = 2, \xi_{u_1,1}^{(s_2)} = 2, \xi_{u_1,1}^{(s_1 s_2 s_1 s_2)} = 0, \\ \xi_{u_1,z_2}^{(s_2)} &= 0, \xi_{u_1,z_2}^{(s_1 s_2 s_1 s_2)} = 2, \xi_{u_0,1}^{(s_1 s_2)} = 1. \end{aligned}$$

$G$  of type  $G_2$ .

$$\mathbf{s}_{u_6} = (1), \mathbf{s}_{u_3} = (s_2), \mathbf{s}_{u_2} = (s_1), \mathbf{s}_{u_1} = \{(s_1 s_2 s_1 s_2 s_1 s_2), (s_1 s_2 s_1 s_2)\}, \mathbf{s}_{u_0} = (s_1 s_2).$$

$$\begin{aligned} \xi_{u_6,1}^1 &= 12, \xi_{u_3,1}^{(s_2)} = 2, \xi_{u_2,1}^{(s_1)} = 2, \xi_{u_1,1}^{(s_1 s_2 s_1 s_2 s_1 s_2)} = -3, \xi_{u_1,z_2}^{(s_1 s_2 s_1 s_2 s_1 s_2)} = 3, \\ \xi_{u_1,z_3}^{(s_1 s_2 s_1 s_2 s_1 s_2)} &= 0, \xi_{u_1,1}^{(s_1 s_2 s_1 s_2)} = 2, \xi_{u_1,z_2}^{(s_1 s_2 s_1 s_2)} = 0, \xi_{u_1,z_3}^{(s_1 s_2 s_1 s_2)} = 2, \xi_{u_0,1}^{(s_1 s_2)} = 1. \end{aligned}$$

$G$  of type  $B_3$ .

$$\begin{aligned} \mathbf{s}_{u_9} &= (1), \mathbf{s}_{u_5} = (s_1), \mathbf{s}_{u_4} = \{(s_3), (s_2 s_3 s_2 s_3)\}, \mathbf{s}_{u_3} = (s_1 s_3), \\ \mathbf{s}_{u_2} &= \{(s_1 s_2), (w_0)\}, \mathbf{s}_{u_1} = \{(s_2 s_3), (s_2 s_3 s_1 s_2 s_3)\}, \mathbf{s}_{u_0} = (s_1 s_2 s_3). \end{aligned}$$

$$\begin{aligned} \xi_{u_9,1}^1 &= 48, \xi_{u_5,1}^{(s_1)} = 4, \xi_{u_4,1}^{(s_2 s_3 s_2 s_3)} = 0, \xi_{u_4,z_2}^{(s_2 s_3 s_2 s_3)} = 4, \xi_{u_4,1}^{(s_3)} = 8, \\ \xi_{u_4,1}^{(s_3)} &= 0, \xi_{u_3,1}^{(s_1 s_3)} = 2, \xi_{u_2,1}^{(w_0)} = 0, \xi_{u_2,z_2}^{(w_0)} = 6 \\ \xi_{u_2,1}^{(s_1 s_2)} &= 2, \xi_{u_2,z_2}^{(s_1 s_2)} = 0, \xi_{u_1,1}^{(s_2 s_3)} = 2, \xi_{u_1,z_2}^{(s_2 s_3)} = 0, \\ \xi_{u_1,1}^{(s_2 s_3 s_1 s_2 s_3)} &= 0, \xi_{u_1,z_2}^{(s_2 s_3 s_1 s_2 s_3)} = 2, \xi_{u_0,1}^{(s_1 s_2 s_3)} = 1. \end{aligned}$$

$G$  of type  $C_3$ .

$$\begin{aligned} \mathbf{s}_{u_9} &= (1), \mathbf{s}_{u_6} = (s_3), \mathbf{s}_{u_4} = \{(s_1), (s_2 s_3 s_2 s_3)\}, \mathbf{s}_{u_3} = (s_1 s_3), \\ \mathbf{s}_{u_2'} &= (s_1 s_2), \mathbf{s}_{u_2''} = (s_2 s_3), \mathbf{s}_{u_1} = \{(s_2 s_3 s_1 s_2 s_3), w_0\}, \mathbf{s}_{u_0} = (s_1 s_2 s_3). \end{aligned}$$

$$\begin{aligned} \xi_{u_9,1}^1 &= 48, \xi_{u_6,1}^{(s_3)} = 8, \xi_{u_4,1}^{(s_2 s_3 s_2 s_3)} = 0, \xi_{u_4,z_2}^{(s_2 s_3 s_2 s_3)} = 4, \\ \xi_{u_4,1}^{(s_1)} &= 4, \xi_{u_4,1}^{(s_1)} = 0, \xi_{u_3,1}^{(s_1 s_3)} = 2, \xi_{u_2',1}^{(s_1 s_2)} = 2, \xi_{u_2'',1}^{(s_2 s_3)} = 2, \\ \xi_{u_1,1}^{(s_2 s_3 s_1 s_2 s_3)} &= 1, \xi_{u_1,z_2}^{(s_2 s_3 s_1 s_2 s_3)} = 1, \xi_{u_1,1}^{(w_0)} = -3, \xi_{u_1,z_2}^{(w_0)} = 3, \xi_{u_0,1}^{(s_1 s_2 s_3)} = 1. \end{aligned}$$

**9.** For any unipotent element  $u \in G$  let  $n_u$  be the number of isomorphism classes of irreducible representations of  $\bar{Z}(u)$  which appear in the Springer correspondence for  $G$ . Consider the following properties of  $G$ :

$$(a) \quad \mathbf{W} = \sqcup_u \mathbf{s}_u$$

( $u$  runs over a set of representatives for the unipotent classes in  $G$ ); for any unipotent element  $u \in G$ ,

$$(b) \quad |\mathbf{s}_u| = n_u.$$

By the results in §8, (a),(b) hold if  $G$  has rank  $\leq 3$ . We will show elsewhere that (a),(b) hold if  $G$  is of type  $A$ . We expect that (a),(b) hold in general. The equality  $\mathbf{W} = \sqcup_u \mathbf{s}_u$  is clear since for a regular unipotent  $u$  and any  $w$  we have  $\xi_{u,1}^w = 1$ . Consider also the following property of  $G$ : for any  $g \in G$ ,  $w \in \mathbf{W}$ ,

$$(c) \quad \begin{array}{l} \xi_{g,1}^w \text{ is equal to the trace of } w \text{ on the Springer representation} \\ \text{of } \mathbf{W} \text{ on } \oplus_i H^{2i}(\mathcal{B}_g, \bar{\mathbf{Q}}_l). \end{array}$$

Again (c) holds if  $G$  is of type  $A$  and in the examples in §8; we expect that it holds in general. Note that in (c) one can ask whether for any  $z$ ,  $\xi_{g,z}^w$  is equal to the trace of  $wz$  on the Springer representation of  $\mathbf{W} \times \bar{Z}(g)$  on  $\oplus_i H^{2i}(\mathcal{B}_g, \bar{\mathbf{Q}}_l)$ ; but such an equality is not true in general for  $z \neq 1$  (for example for  $G$  of type  $B_2$ ).

**10.** In this subsection we assume that  $\mathbf{k} = \mathbf{C}$ . We show how the method of [KL, 9.1] extends to the case of symmetric spaces.

Let  $K$  be a subgroup of  $G$  which is the fixed point set of an involution  $\theta : G \rightarrow G$ . Let  $\mathfrak{g}, \mathfrak{k}$  be the Lie algebras of  $G, K$ . Let  $\mathfrak{p}$  be the  $(-1)$ -eigenspace of  $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ . Let  $\mathfrak{p}_{nil}$  be the set of elements  $x \in \mathfrak{p}$  which are nilpotent in  $\mathfrak{g}$ . Note that  $K$  acts naturally on  $\mathfrak{p}_{nil}$  with finitely many orbits. Let  $\mathcal{T}$  be the variety of all tori  $T$  in  $G$  such that  $\theta(t) = t^{-1}$  for all  $t \in T$  and such that  $T$  has maximum possible dimension. It is known that  $K$  acts transitively on  $\mathcal{T}$  by conjugation. For  $T \in \mathcal{T}$  let  $\mathcal{W}_T$  be the normalizer of  $T$  in  $K$  modulo the centralizer of  $T$  in  $K$ . Let  $\underline{\mathcal{W}}$  be the set of conjugacy classes in the finite group  $\mathcal{W}_T$ ; this is independent of the choice of  $T$ .

Let  $\Phi = \mathbf{C}((\epsilon))$ ,  $A = \mathbf{C}[[\epsilon]]$  where  $\epsilon$  is an indeterminate. Let  $\Phi'$  be an algebraic closure of  $\Phi$ . Then the groups  $G(\Phi), K(\Phi)$  are well defined and the set  $\mathcal{T}(\Phi)$  of  $\Phi$ -points of  $\mathcal{T}$  is well defined. Let  $\mathfrak{p}_\Phi = \Phi \otimes \mathfrak{p}$ ,  $\mathfrak{p}_A = A \otimes \mathfrak{p}$ .

The group  $K(\Phi)$  acts naturally by conjugation on  $\mathcal{T}(\Phi)$ ; as in [KL, §1, Lemma 2] we see that the set of  $K(\Phi)$ -orbits on  $\mathcal{T}(\Phi)$  is naturally in 1-1 correspondence with the set  $\underline{\mathcal{W}}$ . For  $\gamma \in \underline{\mathcal{W}}$  let  $\mathcal{O}_\gamma$  be the  $K(\Phi)$ -orbit on  $\mathcal{T}(\Phi)$  corresponding to  $\gamma$ .

An element  $\xi \in \mathfrak{p}_\Phi$  is said to be "regular semisimple" if there is a unique  $T' \in \mathcal{T}(\Phi')$  such that  $\xi \in \text{Lie}(T')$ ; we then set  $T'_\xi = T'$  and we have necessarily  $T'_\xi \in \mathcal{T}(\Phi)$ .



Let  $N \in \mathfrak{p}_{nil}$ . We consider the subset  $N + \epsilon \mathfrak{p}_A$  of  $\mathfrak{p}_\Phi$ . As in [KL, 9.1] there is a unique element  $\gamma \in \underline{\mathcal{W}}$  such that the following holds: there exists a "Zariski open dense" subset  $V$  of  $N + \epsilon \mathfrak{p}_A$  such that for any  $\xi \in V$ ,  $\xi$  is "regular semisimple" and  $T'_\xi \in \mathcal{O}_\gamma$ . Note that  $N \mapsto \gamma$  is constant on  $K$ -orbits hence it defines a map  $\Psi$  from the set of  $K$ -orbits on  $\mathfrak{p}_{nil}$  to  $\underline{\mathcal{W}}$ . (In the case where  $G = H \times H$  where  $H$  is a connected reductive group,  $K$  is the diagonal in  $H \times H$  and  $\theta(a, b) = (b, a)$ , the map  $\Psi$  reduces to the map defined in [KL, 9.1].)

One can show that if  $(G, K) = (GL_{2n}(\mathbf{C}), Sp_{2n}(\mathbf{C}))$ , then  $\Psi$  is a bijection. Note that for general  $(G, K)$ ,  $\Psi$  is neither injective nor surjective.

## REFERENCES

- [CR] B.Chang and R.Ree, *The characters of  $G_2(q)$* , Istituto Naz.di Alta Mat. Symposia Math. **XIII** (1974), 395-413.
- [DL] P.Deligne and G.Lusztig, *Representations of reductive groups over finite fields*, Ann.Math. **103** (1976), 103-161.
- [GP] M.Geck and G.Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, Clarendon Press Oxford, 2000.
- [KL] D.Kazhdan and G.Lusztig, *Fixed point varieties on affine flag manifolds*, Isr.J.Math. **62** (1988), 129-168.
- [L1] G.Lusztig, *On the reflection representation of a finite Chevalley group*, Representation theory of Lie groups, LMS Lect.Notes Ser.34, Cambridge U.Press, 1979, pp. 325-337.
- [L2] G.Lusztig, *Unipotent representations of a finite Chevalley group of type  $E_8$* , Quart.J.Math. **30** (1979), 315-338.
- [L3] G.Lusztig, *Character sheaves, I*, Adv.in Math. **56** (1985), 193-237.
- [L4] G.Lusztig, *On the character values of finite Chevalley groups at unipotent elements*, J.Alg. **104** (1986), 146-194.
- [L5] G.Lusztig, *Notes on unipotent classes*, Asian J.Math. **1** (1997), 194-207.
- [Sh] T.Shoji, *On the Green polynomials of Chevalley groups of type  $F_4$* , Comm.in Alg. **10** (1982), 505-543.
- [S1] N.Spaltenstein, *Polynomials over local fields, nilpotent orbits and conjugacy classes in Weyl groups*, Astérisque **168** (1988), 191-217.
- [S2] N.Spaltenstein, *On the Kazhdan-Lusztig map for exceptional Lie algebras*, Adv.Math **83** (1990), 48-74.

DEPARTMENT OF MATHEMATICS, M.I.T., CAMBRIDGE, MA 02139